Distribution & Interpolation Spaces – Exercise sheet 1

Exercise 1. Let $f, g \in C_c(\mathbb{R}^n)$ and consider the convolution fuction defined as

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy.$$

Show that $f * g$ is still continuous, with compact support and it is a $L^1(\mathbb{R}^n)$ function satisfying

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}. \tag{1}$$

Exercise 2. Let $f \in L^1(\mathbb{R}^n)$ and for any $k \in \mathbb{N}$ consider the truncated functions

$$f_k(x) = \begin{cases} k & \text{if } f(x) > k \\ f(x) & \text{if } |f(x)| \leq k \\ -k & \text{if } f(x) < -k \end{cases}$$

Show that, given an arbitrary $g \in L^1(\mathbb{R}^n)$, the sequence $(f_k * g)_k$ strongly converges to $f * g$ in $L^1(\mathbb{R}^n)$.

Exercise 3.

i) Let $f, g \in L^1(\mathbb{R}^n)$. Show that the convolution formula

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

defines a function $f * g$ in $L^1(\mathbb{R}^n)$ for which estimate (1) of Exercise 1 still holds.

ii) Let now $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1 + \frac{1}{r} \geq 1$. Show that $f * g \in L^r(\mathbb{R}^n)$ and satisfies

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}.$$

Exercise 4. Let $f \in C(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Show that $f * g \in C(\mathbb{R}^n)$. Let then $f \in C^1(\mathbb{R}^n)$ be such that $Df \in L^{p'}(\mathbb{R}^n)$. Prove that, in this case, $f * g \in C^1(\mathbb{R}^n)$ and for every $1 \leq i \leq n$ one has

$$\frac{\partial(f * g)}{\partial x_i} = \frac{\partial f}{\partial x_i} * g.$$

Exercise 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function. We denote by $f_\varepsilon = f * \rho_\varepsilon$ the sequence obtained by convolution between f and the standards *mollifiers* functions $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^n, [0, \infty))$

$$\int_{\mathbb{R}^n} \rho_\varepsilon(x)dx = 1 \qquad \text{and} \qquad spt\,\rho_\varepsilon \subseteq B(0, \varepsilon).$$

Prove the following statements.

i) If f is uniformly continuous, then $f_\varepsilon \to f$ uniformly.

ii) If $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^n)$, then f_ε strongly converges to f in $L^p(\mathbb{R}^n)$.

iii) If $p \in [1, \infty)$, then $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.